# Approximation of Integrals

**The midpoint rule** To approximate $\int_a^b f(x)dx$ by the *midpoint formula* with $n$ subdivisions:
1. We define the *subinterval length* $\Delta x = \frac{b-a}{n}$ and
2. Create the subintervals from $a$ to $a+\Delta x$, from $a+\Delta x$ to $a+2\Delta x$ etc.
3. For each subinterval we take the number $x_i$ which is the middle of the interval and calculate $f(x_i)$
4. We multiply each function value $f(x_i)$ by $\Delta x$ and add the results [or add all the values $f(x_i)$ and multiply the sum by $\Delta x$]

Our approximation can be represented as
$M_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x$
**Error Bound** The error of this approximation is *no greater than* $\frac{B_2(b-a)^3}{24n^2}$ with $B_2 = \text{Max}\,|f''(x)|$ (on the interval $a$ to $b$)
The approximation amounts to replacing each piece of the curve (over one subinterval) by a horizontal line segment (a constant value) at the height of the curve in the middle of the subinterval

**The trapezoid rule** To approximate $\int_a^b f(x)dx$ by the *trapezoid formula* with $n$ subdivisions:
1. We define the *subinterval length* $\Delta x = \frac{b-a}{n}$ and
2. Create the subintervals from $a$ to $a+\Delta x$, from $a+\Delta x$ to $a+2\Delta x$ etc.
3. We take the values at the ends of the intervals: $x_0 = a, x_1 = a+\Delta x, \cdots x_n = a+n\Delta x = b$ and calculate $f(x_i)$ for each.
4. We add *one* copy each of $f(x_0)$ and $f(x_n)$ (the endpoint values) and *two* times each of the other values, and multiply the sum by $\frac{\Delta x}{2}$

Our approximation can be represented as
$T_n = (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))\frac{\Delta x}{2}$
(This is the average of the left-hand sum and the right-hand sum we discussed when defining the integral)
**Error Bound** The error of this approximation is *no greater than* $\frac{B_2(b-a)^3}{12n^2}$ with $B_2 = \text{Max}\,|f''(x)|$ (on the interval $a$ to $b$)
It allows a larger error than the midpoint rule (for a particular value of $n$ - but since we do not have to find the midpoints of the intervals it often involves less calculation (for a particular value of $n$).
The approximation amounts to replacing each piece of the curve (over one subinterval) by a straight line segment which gives the same values as the function at the start and end points of the subinterval.

**Simpson's rule** To approximate $\int_a^b f(x)dx$ by *Simpson's rule* with $n$ subdivisions, $n$ must be an *even* number:
1. We define the *subinterval length* $\Delta x = \frac{b-a}{n}$ and
2. Create the subintervals from $a$ to $a+\Delta x$, from $a+\Delta x$ to $a+2\Delta x$ etc.
3. We take the values at the ends of the intervals: $x_0 = a, x_1 = a+\Delta x, \cdots x_n = a+n\Delta x = b$ and calculate $f(x_i)$ for each.
4. We add *one* copy each of $f(x_0)$ and $f(x_n)$ (the endpoint values) and *four* times each of the values at *odd*-numbered points and *two* times each of the values at the *even*-numbered points, and multiply the sum by $\frac{\Delta x}{3}$

Our approximation can be represented as
$S_n = (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))\frac{\Delta x}{3}$
**Error Bound** The error of this approximation is *no greater than* $\frac{B_4(b-a)^5}{180n^4}$ with $B_4 = \text{Max}\left|f^{(4)}(x)\right|$ (largest absolute value of fourth derivative on the interval $a$ to $b$)
The approximation amounts to replacing each piece of the curve over two subintervals by a segment of a parabola which gives the same values as the function at the three function points used.