# Answers to Exam 4 Practice Problems

1. (a) $\sqrt{1/2}$.

   (b) 0.

   (c) $(e^2)^2 = e^4$. Use the fact that if $f$ is a continuous function and $\{a_n\}$ is a sequence that converges to $L$, then $f(a_n) \to f(L)$, together with your knowledge of indeterminate forms such as $1^\infty$ and L'Hôpital's rule.

   (d) 0. Can use Squeeze Principle (trap the terms between 0 and $\frac{2}{n-1}$) or just simplify to $\lim\limits_{n\to\infty} \dfrac{2}{(n-1)!}$.

   (e) 0.

   (f) $\infty$.

   (g) 1. This limit has an indeterminate form. Set $y = \lim\limits_{x\to\infty} x^{1/x}$. Then

   $$\ln y = \lim_{x\to\infty} (1/x) \cdot \ln x = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

   by L'Hôpital's rule. Thus, $y = 1$.

   (h) 0. By the Squeeze Principle, since

   $$\frac{-1}{n^4+3n+2} \leq \frac{\cos n}{n^4+3n+2} \leq \frac{1}{n^4+3n+2}$$

   and

   $$\lim_{n\to\infty} \frac{-1}{n^4+3n+2} = \lim_{n\to\infty} \frac{1}{n^4+3n+2} = 0,$$

   it must follow that $\lim\limits_{n\to\infty} \dfrac{\cos n}{n^4+3n+2} = 0$, as well.

   (i) 0. Use L'Hôpital's rule twice to determine $\lim\limits_{x\to\infty} \dfrac{x^2+3x}{2^x}$.

2. Yes. The sequence is clearly nondecreasing (in fact, it is increasing). Moreover, it is bounded below by 0 and above by .3. Thus, being monotonic (increasing) and bounded above, the sequence converges.

3. $\displaystyle\sum_{k=4}^{\infty} \frac{7}{6^k} = \frac{7}{6^4} + \frac{7}{6^5} + \frac{7}{6^6} + \ldots$ is a geometric series with $a = \frac{7}{6^4}$ and $r = 1/6$, so the sum is $\frac{7}{6^4}\left(\frac{1}{1-1/6}\right) = \frac{7}{1080}$.

4. No. Since $\{\frac{8k}{5k+2}\}$ converges to 8/5, which is not 0, the series diverges by the *nth* term divergence test.

5. Yes. The series $\displaystyle\sum_{j=1}^{\infty} e^{-j}$ is a geometric series, with $|r| = e^{-1} < 1$. The sum is $\frac{1/e}{1-1/e} = \frac{1}{e-1}$.

6. One option is to use the Ratio Test. We find

   $$\lim_{k\to\infty} \left(\frac{6(k+1)}{9^{k+1}+11}\right)\left(\frac{9^k+11}{6k}\right) = \lim_{k\to\infty} \left(\frac{6(k+1)}{6k}\right)\left(\frac{9^k+11}{9\cdot 9^k+11}\right) = 1/9.$$

   Since $1/9 < 1$, the series converges by the Ratio Test.

7. Let $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5+3^k}$ be an infinite series.

(a) 0.1250000000, 0.07142857143, 0.03125000000, 0.01162790698

(b) 0.1250000000, 0.1964285714, 0.2276785714, 0.2393064784

(c) Yes. Clearly, $\frac{1}{5+3^k} \leq \frac{1}{3^k}$ for all $k$. Also the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges being a geometric series with $|r| = 1/3 < 1$. Thus, by the Comparison Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5+3^k}$ converges.

(d) i. Can you find the exact sum of this series? No.

ii. Estimate the sum. I'll use $S_{10} = .2453773644$.

iii. How good is the estimate given above? We know that

$$R_{10} = \sum_{k=11}^{\infty} \frac{1}{5+3^k} \leq \sum_{k=11}^{\infty} \frac{1}{3^k} = \frac{(1/3)^{11}}{1-(1/3)} \approx .0000085.$$

iv. Find $n$ to estimate the sum within .001. Then compute $S_n$ for this $n$.
We could use $S_{10}$ from part iii, clearly accurate enough, or solve for $n$ in the inequality:

$$R_n \leq \frac{(1/3)^{n+1}}{1-(1/3)} < .001.$$

v. Find an upper and lower bound for the sum.

For a lower bound, use $S_10$. For an upper bound, we note that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5+3^k} < \sum_{k=1}^{\infty} \frac{1}{3^k}$.

And $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1/3}{1-(1/3)} = 1/2$. So an upper bound is 1/2.

8. (a) The series $\displaystyle\sum_{k=2}^{\infty} \frac{4k}{5k^2-3}$ diverges, as we now show. It is clear that $\frac{4k}{5k^2-3} > \frac{4k}{5k^2} = \frac{4}{5k}$ for all $k \geq 2$ Moreover, the series $\displaystyle\sum_{k=2}^{\infty} \frac{4}{5k}$ diverges, being a mutiple of the divergent harmonic series. Hence by the Comparison Test, the series $\displaystyle\sum_{k=2}^{\infty} \frac{4k}{5k^2-3}$ diverges.

(b) The series can be written as the difference of two convergent geometric series:

$$\sum_{k=1}^{\infty} \frac{1-2^k}{5^k} = \sum_{k=1}^{\infty} \frac{1}{5^k} - \sum_{k=1}^{\infty} \frac{2^k}{5^k} = \frac{1}{4} - \frac{2}{3} = \frac{-5}{12}.$$

(c) We apply the ratio test. Since

$$\lim_{m\to\infty} \frac{\frac{(m+1)\cdot 2^{m+1}}{(m+1)!}}{\frac{m\cdot 2^m}{m!}} = \lim_{m\to\infty} \frac{(m+1)\cdot 2^{m+1}\cdot m!}{m\cdot 2^m\cdot (m+1)!} = \lim_{m\to\infty} \frac{2}{m} = 0,$$

so $L < 1$, the series converges. Ignore the upper bound on this one–just approximate the series.

(d) (Error in problem statement: sum should start at $j = 2$.) Since

$$\int_2^\infty \frac{dx}{x(\ln x)^3} = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^3} = \lim_{t\to\infty} \frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2},$$

the infinite series $\displaystyle\sum_{j=2}^{\infty} \frac{1}{j(\ln j)^3}$ converges, as well, by the integral test. Further, the sum is bounded as follows:

$$\frac{1}{2(\ln 2)^2} \leq \sum_{j=1}^{\infty} \frac{1}{j(\ln j)^3} \leq \frac{1}{2(\ln 2)^2} + \frac{1}{2(\ln 2)^3}.$$

(e) This is a telescoping series: $S_n = \arctan(n+1) + \arctan(n+2) - \arctan 0 - \arctan 1$, and

$$\lim_{n\to\infty} S_n = \frac{\pi}{2} + \frac{\pi}{2} - 0 - \frac{\pi}{4} = \frac{3\pi}{4}.$$

9. Let $\displaystyle\sum_{k=0}^{\infty} \frac{k+1}{(k^2+2k+2)^2}$ be an infinite series.

(a)

(b)

(c) The series converges. We can use either the integral test or comparison to $\sum_{k=1}^{\infty} \frac{2k}{k^4}$.

(d) i. Can you find the exact sum of this series? No.

ii. Estimate the sum with a partial sum, e.g. $S_5 \approx .3856$.

iii. How good is the estimate given above? We know that

$$R_n = \sum_{k=n+1}^{\infty} \frac{k+1}{(k^2+2k+2)^2} \leq \int_n^\infty \frac{x+1}{(x^2+2x+2)^2} dx$$

by the integral test, so $R_5 = \sum_{k=6}^{\infty} \frac{k+1}{(k^2+2k+2)^2} \leq \int_5^\infty \frac{x+1}{(x^2+2x+2)^2} dx \approx .0135$

iv. To find $n$ to estimate the sum within .001, we'll find an $n$ such that

$$\int_n^\infty \frac{x+1}{(x^2+2x+2)^2} dx = \frac{1}{2(n^2+2n+2)} < .001,$$

then compute $S_n$ for this $n$. Try $n = 22$ and $S_{22}$.

v. An upper and lower bound for the sum are easier to find via the integral test, since

$$\int_0^\infty \frac{x+1}{(x^2+2x+2)^2} dx \leq \sum_{k=0}^{\infty} \frac{k+1}{(k^2+2k+2)^2} \leq \int_0^\infty \frac{x+1}{(x^2+2x+2)^2} dx + 1/4.$$

This bounds the sum between 1/4 and 1/2.

10. Determine the type of convergence.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{k^3+3^k}$ This series diverges. Since the terms of the series do not converge to 0, the series diverges. Note that for $k$ even, the terms tend to 1, and for $k$ odd, the terms tend to -1.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)2^k}$ This series is absolutely convergent. Consider the series $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{(k+1)2^k} \right| = \sum_{k=1}^{\infty} \frac{1}{(k+1)2^k}$. We compare this series to $\sum_{k=1}^{\infty} \frac{1}{2^k}$. We note that $\frac{1}{(k+1)2^k} \leq \frac{1}{2^k}$ for all $k$. The series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, being a geometric series with $|r| = 1/2 < 1$. Hence, by the Comparison Test, the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)2^k}$ converges. Since $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{(k+1)2^k} \right|$ converges, the original series converges absolutely.

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}}$ This series converges conditionally, since $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ is a divergent series and $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}}$ converges by the alternating series test.

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4+k+2}$ This series is absolutely convergent. Compare the positive term (absolute value) series to $\sum_{k=1}^{\infty} \frac{1}{k^4}$. (The formal argument would be like (b).)

11. Consider the series: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}}$.

(a) Show that the series converges.

This series is convergent by the AST. We see that the terms $a_k = \frac{1}{k^{3/2}}$ approach 0 and $\{\frac{1}{k^{3/2}}\}$ is a decreasing sequence. Hence the series converges.

(b) Find an upper and lower bound for the sum of the series.

We see that $S_6 = 1.828487581$ and $S_7 = 1.882482506$. Thus, $S_6$ is a lower bound, and $S_7$ is an upper bound.

(c) To estimate the sum of the series within .001, we need an $n$ such that $c_{n+1} < .001$ or $\frac{1}{(n+1)^{3/2}} < .001$. Any $n \geq 100$ should work (check it). Use $S_{100}$.

12. Find the radius of convergence and the interval of convergence of the following series.

(a) $\sum_{k=1}^{\infty} \frac{(2x)^k}{k^2}$ We use the Ratio Test. Computing the limit we find $\lim_{k\to\infty} \left| \frac{(2x)^{k+1}}{(k+1)^2} \frac{k^2}{(2x)^k} \right| = \lim_{k\to\infty} \left| \frac{(2x)k^2}{(k+1)^2} \right| = |2x|$. For the series to converge, we need the limit to be less than 1. So we need $|2x| < 1$ or $|x| < 1/2$. **Hence, the radius of convergence is 1/2.** Now we check the endpoints. Let $x = 1/2$. Then the series becomes $\sum_{k=1}^{\infty} \frac{(2(1/2))^k}{k^2} = \sum_{k=1}^{\infty} \frac{(1)^k}{k^2}$, which converges being a $p$-series with $p = 2 > 1$. Thus, $x = 1/2$ is in the interval of convergence. Let $x = -1/2$. Then the series becomes $\sum_{k=1}^{\infty} \frac{(2(-1/2))^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$, which converges absolutely (same argument as above). Thus, $x = -1/2$ is in the interval of convergence. **Hence the interval of convergence is** $[-1/2, 1/2]$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(x)^k}{4^k k^{1/2}}$

The interval of convergence is [-4,4).

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(x)^{3k}}{k!}$

The interval of convergence is $(-\infty, \infty)$.

13. (a) This is a geometric power series with ratio $r = 9x^2$ and leading term $a = 9x^2$, so $\displaystyle\sum_{k=1}^{\infty} (3x)^k = \frac{9x^2}{1 - 9x^2}$ on the interval of convergence $(-1/3, 1/3)$.

(b) This is also a geometric power series with $r = a = \frac{5-x}{10}$. So the sum is $\frac{5-x}{5+x}$ for all $x$ in the interval (of convergence) $(-5, 15)$.

(c) This is a hard one, but if we notice that the series $(x-5) + \frac{(x-5)^2}{2} + \frac{(x-5)^3}{3} + ...$ is a term-by-term antiderivative of the series $1 + (x-5) + (x-5)^2 + ...$, we can conclude that the sum is an antiderivative of $\frac{1}{1-(x-5)} = \frac{1}{6-x}$. Thus, the sum is a function of the form $-\ln|6-x| + C$. To find the $C$, we plug the base point $x_0 = 5$ into the series: $(x-5) + \frac{(x-5)^2}{2} + \frac{(x-5)^3}{3} + ... = -\ln|6-x| + C$. We get $C = 0$, so the sum is $-\ln|6-x|$.