**Math 361** **ACTIVITY 3:**Plane separation - some consequences

**Why**

Plane separation and betweenness are the biggest ideas you have encountered in this course that you (probably) did not see in your previous geometry course. Seeing how they fit into a complete set of axioms for Euclidean geometry can give you practice in seeing what is and is not contained in the axioms - how Euclidean geometry works, and how other geometries work. We will use the plane separation axiom to develop some properties of angles that will be needed for angle measure

**LEARNING OBJECTIVES**

1. Work as a team, using the team roles
2. Gain a deeper understanding of the role of plane separation in the axioms for Euclidean geometry
3. Gain skill in reasoning from the axioms in a formal mathematical development
4. Develop some properties of angles needed for our next step.

**CRITERIA**

1. Success in completing the exercises.
2. Success in working as a team
3. Understanding of the ideas involved in plane separation

**RESOURCES**

1. Your text - section 2.6
2. The typed notes on sections 2.2, 2.3, 2.4 and 2.6 [available in Blackboard]
3. Your own notes from class
4. 40 minutes

**PLAN**

1. Select roles, if you have not already done so, and decide how you will carry out steps 2 and 3.
2. Work through the exercises given here - be sure everyone understands all results.
3. Assess the team's work and roles performances and prepare the Reflector's and Recorder's reports including team grade.
4. Be prepared to discuss your results.

**EXERCISE**

We will prove that one big and obvious fact about angles does follow is now built into our system by the plane separation axiom. We will also define the *interior of an angle* which we will need in section 2.5 to develop angle measure (necessary to talk about congruence, among other things)We will be using the geometric structure we have developed so far (axioms I-1 to I-5, D-1 to D-4 and H-1). It will be convenient to use the property of *extendability of a segment* mentioned in class (stated & proved formally here)

**Proposition.** *For any two points $A, B$, there is a point $H$ on $\overleftrightarrow{AB}$ with $H - A - B$*

*Proof.* Let $A, C$ be distinct points. There is a ruler on $\overleftrightarrow{AB}$ for which the coordinate of $A$ is 0 and the coordinate of $B$ (which we call $b$) is positive. There is a point $H$ on $\overleftrightarrow{AB}$ whose coordinate, with this ruler, is $-1$. Now $-1 < 0 < b$ so (Theorem 2.4.3) $H - A - B$, as desired. $\square$

With the plane separation property built into our system, we can define the *interior of an angle*

**Definition.** For any angle $\angle ABC$, the *interior of* $\angle ABC$ is the set of points that are on the $C$ side of $\overleftrightarrow{AB}$ and also on the $A$ side of $\overleftrightarrow{BC}$ (the intersection of the halfplanes — with Kays nice notation we can write: Interior of $\angle ABC = H(A, \overleftrightarrow{BC}) \cap H(C, \overleftrightarrow{BA})$)

Now you will prove that the interior of an angle has some of the basic properties that we expect (from the name):

1. Prove the following (This is Theorem 3 in the notes for section 2.6 and on p. 108 of the text). The proof is based on repeated use of **Theorem 1** and the definition of interior of an angle.

   **Theorem** (rays and segments mostly in the interior of an angle)**.** *Every point of $\overline{AC}$ except $A$ and $C$ is in the interior of $\angle ABC$. If $D$ is in the interior of $\angle ABC$, then every point of $\overrightarrow{BD}$ except $B$ is in the interior of $\angle ABC$.*

2. Prove (by carrying out and tying together the steps given) the following theorem known as the **Crossbar Theorem**. Once again, it's obvious that this should be true in the Euclidean plane. It took people a long time (partly because of undue reverence for Euclid's book) to see that it can't be *proved* without the plane separation axiom – in this case, using the definition of "interior" (based on plane separation) and on Pasch's Theorem (based on plane separation)

   **Theorem.** *If $D$ is a point in the interior of $\angle ABC$, then $\overrightarrow{BD}$ meets $\overline{AC}$ at a point $E$ which is between $A$ and $C$.*

   For our Proof, we consider non-colliner points $A, B, C$ and a point $D$ which is in the interior of $\angle ABC$.Using the proposition proved at the beginning of the exercises, we know there is a point $K$ on $\overleftrightarrow{BC}$ with $K - B - C$ and there is a point $F$ on $\overleftrightarrow{BD}$ with $F - B - D$. Now, your part:

   (a) Draw a diagram of the points, rays, lines specified so far and add $\overline{KC}$ and $\overline{AC}$.

   (b) Show that none of the points $K, A, C$ is on $\overleftrightarrow{BD}$. (our standard type of argument for "this point can't be on that line" — give the details)

   (c) Show that $F$ is on the opposite side of $\overleftrightarrow{BC}$ from $A$. (a "there are only two sides" argument — remember where $D$ is)

   (d) Show that $\overleftrightarrow{BD}$ must meet either $\overline{KA}$ or $\overline{AC}$ at a point other than $K, A$, or $C$. (Pasch is your friend - show how the theorem applies) We'll call the point of intersection $E$

   (e) Show that $E$ (whether it is on $\overline{KC}$ or $\overline{AC}$) cannot be on $\overrightarrow{BF}$, so $E$ is on $\overrightarrow{BD}$. (Use the Z-property – Corollary to Theorem 1 – and Proposition 1 from section 2.4 - give the details)

   (f) Show that $E$ cannot be on $\overline{KA}$ (the Corollary to Theorem 1, again - give the details)

   (g) Put it all together to show $E$ is the point claimed by the theorem.

**SKILL EXERCISES:** (hand in - individually - with this week's assignments)

**Read** Section 2.5 in the text (angle measure and protractor axiom)
**Write 1.** p.111 #13[Pasch & basic incidence thms for intersection with interior of $\overline{AB}$, Thm 1 & Lemma & basic incidence for $D - E - F$] **2.** Prove: For any noncollinear points $A, B, C$, the interior of $\angle ABC$ is a convex set. (basically the definition of "convex" and the definition of interior of an angle - you provide the details to put the pieces together)